The propagation of gravity currents in a circular cross-section channel: experiments and theory

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Supplementary Material

1. Some details for the calculation of φ and Fr, and on the influence of β

The SW calculations use some integrals of the width function of the cross-section, in particular for the area ratio $\varphi = A/A_T$ and for Fr. We used analytical evaluations as follows.

Consider the half-width between $y = 0$ and the curve $y = \tilde{f}(z) = (2rz - z^2)^{1/2}$ for $0 \leqslant z \leqslant h$ $(\leqslant 2r)$ (here lengths are scaled with h_0). The needed integrals are

$$I(h) = \int_0^h (2rz - z^2)^{1/2} dz = \frac{1}{2}\left[(h-r)(2rh - h^2)^{1/2} - r^2 \arcsin(1 - h/r) + \frac{\pi}{2} r^2\right]; \tag{1.1}$$

$$J(h) = \int_0^h (2rz - z^2)^{1/2} z dz = -\frac{1}{3}(2rh - h^2)^{3/2} + rI(h). \tag{1.2}$$

Let $r = H/\beta$, see Figure 1 in the main manuscript. We find that $A = 2I(h)$, $A_T = 2I(H)$, and hence

$$\varphi = I(h)/I(H). \tag{1.3}$$

The following front condition

$$u_N = \frac{1}{R^{1/2}} Fr h_N^{1/2}; \quad Fr^2 = \frac{2(1-\varphi)}{1+\varphi}\left[1 - \varphi + \frac{1}{hA_T}\int_0^h z f(z) dz\right], \tag{1.4}$$

yields

$$Fr^2 = \frac{2(1-\varphi)}{1+\varphi}(1 - \varphi + Q) \tag{1.5}$$

where

$$Q = \frac{J(h)}{hI(H)}. \tag{1.6}$$

It can be shown that, for a given β, Fr is a function of $a = h/H$. The typical behavior is displayed in Figure SM.1. Recall that $\beta = 2$ means that the channel is a full circle (of radius $H/2$), while smaller values of β mean circular sections of radius H/β. For a larger than the values shown in the lines of the figure, the solution of Fr is energetically unacceptable.

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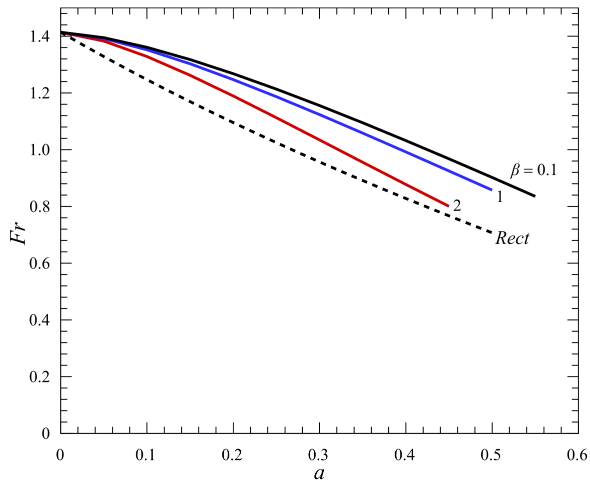

FIGURE SM.1. Fr as a function of $a = h/H$ for various β, circle $f(z) = \left[2(H/\beta)z - z^2\right]^{1/2}$. The dashed line is for the rectangular cross-section. Only values with acceptable physical dissipation are shown.

Figure SM.1 indicates that $Fr(a)$ is quite insensitive to the value of β for $\beta \leqslant 1$. This result points out a useful property: the flow in a semi-circle is well approximated by that in a power-law width function channel, $f(z) = z^{1/2}$. The justification is as follows. For the circle, we use the expansion $(2rz-z^2)^{1/2} = (2rz)^{1/2}(1-z/(4r)+...)$ in the integrands of (1.1)-(1.2). We integrate term by term, and substitute $r = H/\beta$. With an expansion for small β, we find that the leading terms are

$$\varphi = a^{3/2}; \quad Q = \frac{3}{5}a^{3/2}; \tag{1.7}$$

and

$$Fr^2 = \frac{2(1-a^{3/2})}{1+a^{3/2}}(1 - \frac{2}{5}a^{3/2}). \tag{1.8}$$

We see that β cancels out from these first-order approximate expressions. The relative error due to the neglected terms is smaller than 0.15β for φ in (1.7), and about $0.25a^{3/2}\beta$ for Fr in (1.8). The approximate results (1.7) and (1.8) for the circle are the exact results for φ and Fr in a power-law $f(z) = z^{1/2}$ channel. See §4 in Ungarish (2013).

The approximation (1.8) is indeed in very good agreement with the exact calculation of Fr for $\beta \leqslant 1$: in Figure SM.1 the exact line for $\beta = 0.1$ cannot be distinguished from one obtained with (1.8). Furthermore, the differences between the lines $\beta = 1$ and 0.1 is small. The representation of the semi-circular boundary by the power-law $(2rz)^{1/2}$ approximation is essential in the derivation of the self-similar flow results.

Rigorously, the $(2rz-z^2)^{1/2}$ width function is incompatible with t^γ propagation, while $(2rz)^{1/2}$ is (see Zemach & Ungarish, 2013). Since the self-similar current is very thin, $z/r \ll 1$, the approximation is well justified.

2. The relative error in predicting the velocity of the reflected bore and the gravity current front velocity

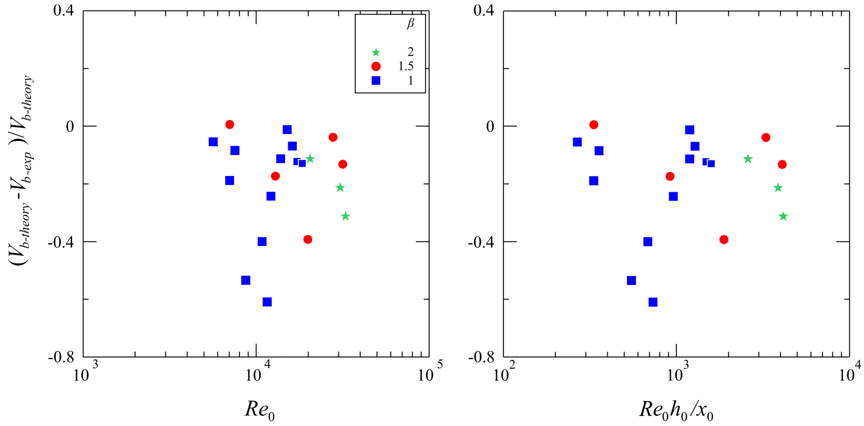

FIGURE SM.2. The relative error in predicting the velocity of the reflected bore in the lock as a function of the initial Reynolds number of the current (left panel) and of the non dimensional group $Re_0(h_0/x_0)$ (right panel).

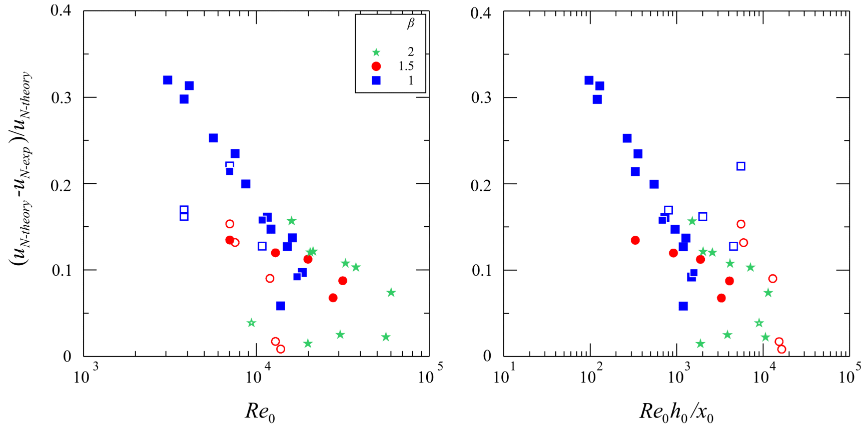

FIGURE SM.3. The relative error in predicting the gravity current front velocity as a function of the initial Reynolds number of the current (left panel) and of the non dimensional group $Re(h_0/x_0)$ (right panel). Filled symbols refer to the long lock experiments, empty symbols refer to the short lock experiments.