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Self-similarity and physics counterpart: a pragmatical approach

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Abstract

We present a pragmatic view of self-similarity, which is indeed rooted in refined mathematical theory and unravelling in multiple approximations, hypotheses, and working approaches, all aimed at solving differential problems by extracting the essence of the results. In developing self-similar solutions, the main concern of the scientist is not the strict adherence to boundary conditions and initial conditions, but rather in discovering their ability to reveal the profound internal harmony that governs many physical processes. The structure of the self-similar variables, almost always of the power type, indicates that some scales of the process are correlated. This correlation is sometimes directly detectable, other times is hidden and originates from the structure of the differential problem itself, and this categorizes the self-similarities of the first and second kind, respectively.

In what follows, after a short introduction of group theory, we provide a step-by-step analysis of two physical problems of fluid mechanics that admit self-similarity solutions of the two kinds listed above. We highlight that we would not have presented these two cases if they had not been experimentally validated, since the more refined and brilliant solution of a differential problem, loses its attractiveness without experimental validation.

1 Introduction

Dimensional Analysis, and the few (practically only one) theorems pertaining to it, derives from the principle of general covariance in physics: all physical laws can be expressed in such a way to be independent of the observer. In the case of quantities, this is a consequence of the evidence that the intrinsic value of quantities (e.g. the height of a person, their weight, the mass of air in a room) must be preserved in all systems of units. For example, if a room contains a mass of air of 58 kg and a smaller room contains 34 kg,

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with a ratio of 58/34=1.7, the ratio between the air mass of the largest room expressed in pounds (58 kg $\rightarrow$ 128 pounds) and the air mass of the smallest room expressed in pounds (34 kg $\rightarrow$ 75.06 pounds) must still be equal to 1.7; in fact 128/75.06 = 1.70 (not including rounding). In this sense, the principle of dimensional homogeneity becomes a natural consequence of the covariance principle, just as the structure of the dimensional equations, in the form of power-functions, is dictated by this covariance principle.

The fundamental idea is that if the intrinsic value of quantities is invariant, in the sense that it does not depend on the system of units of measurement, it must be expressed in such a way that the ratio to the value of another quantity of the exact same nature is the same from one system of units to another. As a consequence, once certain quantities have been identified as *fundamental quantities* (we do not discuss here the conventionality about the selection of fundamental quantities, both in terms of their number and their nature), any *dependent quantity* must be a power-function of these fundamental quantities, and the systems of units are related through a group of *scaling transformations* (see later about groups).

Reasoning in the other way round, if we can prove that the mathematical description of a physical process (for example, a differential problem) is susceptible to invariance within a group of transformations, it can always be expressed in such a way to involve variables that are invariant within the same group. The advantage follows because the number of such variables is smaller than the number of starting variables, and equal to the initial number of variable minus the number of parameters of the transformations. We have thus introduced the concept of transformation, which underlies the methods adopted for the detailed analysis of the properties of many differential problems, including self-similarity of the first and second kind.

In this regard, let us consider what happens for a scaling transformation applied to an ordinary differential equation (ODE). Suppose that our problem is described by the following first-order ODE:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = F(x, y), \tag{1}$$

and assume that it is invariant within the following scaling transformations:

$$\begin{cases} x^* = \alpha x, \\ y^* = \beta y. \end{cases} \tag{2}$$

It can be shown (Bluman & Kumei, 1989) that if β is independent of α, then eq.(1) is with separable variables of the type

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a\frac{y}{x}, \tag{3}$$

where a is a constant. In other words, the separation of variables necessarily requires the scaling transformations to be a two-parameter independent α and β.

On the contrary, if $\beta = \beta(\alpha)$ it is immediate to verify that eq.(1) reduces to

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^{k-1} G(y/x^k), \tag{4}$$

where k is a constant, which is invariant under the one-parameter transformation

$$\begin{cases} x^* = \alpha x, \\ y^* = \alpha^k y. \end{cases} \tag{5}$$

Ultimately, a scaling that guarantees invariance of an ODE introduces variables of the form y/x^k.

2 The Lie groups

Indeed the previous reasoning belong to the group theory, with particular emphasis on the *Lie groups theory*. In what follows, we will use an extremely simplified version of the theory, which will be useful for understanding the fundamentals of the technique and for deriving results of practical application.

Let $x = (x_1, x_2, \ldots, x_n)$ be a vector in a sub-domain of $\mathbb{R}^n$, and let

$$\mathbf{x}^* = X(\mathbf{x}, \epsilon) \tag{6}$$

be a set of one-parameter transformations, where ϵ is the parameter. The set of transformations forms a group iff: being $\mathbf{x}^* = X(\mathbf{x}, \epsilon_1)$ and $\mathbf{x}^{**} = X(\mathbf{x}, \epsilon_2)$, then $\mathbf{x}^{**} = X(\mathbf{x}, \phi(\epsilon_1, \epsilon_2))$, where $\phi(\ldots)$ is the composition rule of the parameter; in addition, $\mathbf{x}^* = \mathbf{x}$ if $X(\mathbf{x}, e) = \mathbf{x}$, being e the neutral element. For example, if we consider the transformation (5), the function $\phi(\epsilon_1, \epsilon_2) = \epsilon_1 \epsilon_2$ and $e = 1$. The transformation (6) can be expanded about $\epsilon = 0$ obtaining

$$\mathbf{x}^* = \mathbf{x} + \left(\left. \frac{\partial X}{\partial \epsilon} \right|_{\epsilon=0} \right) \epsilon + O(\epsilon^2), \tag{7}$$

or

$$\mathbf{x}^* \approx \mathbf{x} + \xi(\mathbf{x})\epsilon, \quad \text{with} \quad \xi(\mathbf{x}) = \left. \frac{\partial X}{\partial \epsilon} \right|_{\epsilon=0}, \tag{8}$$

which is named the *infinitesimal transformation of the Lie group*. Essentially, it is the lowest approximation of the general transformation for small values of the parameter ϵ.

We also define the *infinitesimal generator of the one-parameter Lie group* the operator

$$\chi(\ldots) = \sum \xi_i \frac{\partial(\ldots)}{\partial x_i}. \tag{9}$$

The invariance in a Lie group of a given differential function requires $F(\mathbf{x}) = F(\mathbf{x}^*)$ or $F(\mathbf{x}) = F(X(\mathbf{x}, \epsilon))$, which can approximated as

$$F(x_i^*) = F(x_i) + \xi_i \frac{\partial F}{\partial x_i} \epsilon + \ldots. \tag{10}$$

So, the invariance reduces to

$$\xi_i \frac{\partial F}{\partial x_i} = 0, \tag{11}$$

which can be interpreted as the invariance of the function ($\mathrm{d}F = 0$) on the characteristic curve represented by

$$\frac{\mathrm{d}x_1}{\xi_1} = \frac{\mathrm{d}x_2}{\xi_2} = \ldots = \frac{\mathrm{d}x_n}{\xi_n}. \tag{12}$$

Examples of one-parameter Lie groups of transformations are *Translations* in the plane,

$$\begin{cases} x^* = x + \epsilon, \\ y^* = y, \quad \epsilon \in \mathbb{R}, \end{cases} \tag{13}$$

with identity element $e = 0$ and composition rule of the parameter $\phi(\epsilon_1, \epsilon_2) = \epsilon_1 + \epsilon_2$; *Scalings* in the plane,

$$\begin{cases} x^* = \alpha x, \\ y^* = \alpha^2 y, \quad 0 < \alpha < \infty, \end{cases} \tag{14}$$

with identity element $e = 1$ and $\phi(\epsilon_1, \epsilon_2) = \epsilon_1 \epsilon_2$. Scalings in the plane can also be expressed as

$$\begin{cases} x^* = (1+\epsilon)x, \\ y^* = (1+\epsilon)^2 y, \quad -1 < \epsilon < \infty, \end{cases} \tag{15}$$

with identity element $e = 0$ and $\phi(\epsilon_1, \epsilon_2) = \epsilon_1 + \epsilon_2 + \epsilon_1\epsilon_2$. A third relevant group is *Rotations* in the plane, expressed as

$$\begin{cases} x^* = x \cos\epsilon - y \sin\epsilon, \\ y^* = x \sin\epsilon + y \cos\epsilon. \end{cases} \tag{16}$$

A number of groups of transformations have been encoded, which are useful for applying a variety of calculation techniques.

The next steps in this analysis become progressively more difficult and of limited interest to non-mathematicians. What emerges, however, is Lie's idea who extended to differential equations the methods that Galois had developed for algebraic equations. In essence, Lie devised a method to determine whether a differential problem is invariant in some group; if so, it follows that it is possible to reduce the ODEs by one order, possibly in cascade up to first order and then to the solution by quadrature.

Of particular interest is the case where a partial derivative equation (PDE) is involved that, with boundary and initial conditions (where one independent variable is time), is invariant within a group of transformations: for a one-parameter transformation, it is possible to reduce the number of variables by

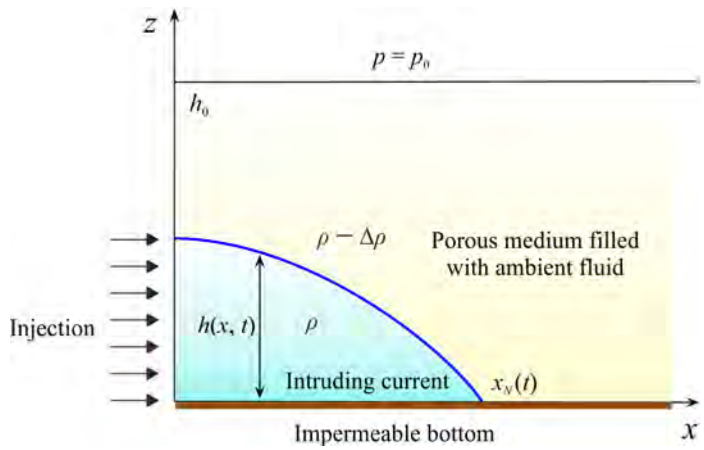

Figure 1: Schematic of the two-dimensional gravity current.

one unit, often reducing the PDE to an ODE. This is not the solution to the problem, but a big step forward since the integration of an ODE is much simpler than the integration of a PDE; in lucky cases, the ODE also admits an analytical solution.

Let us leave aside all the theory and techniques that allow such a transformation to be identified: if the transformation is identified (in most cases it is a scaling, which brings to power-type functions, although non-power-type self-similar solutions are possible, see § 6 in Barenblatt, 1996), then let us delve into how to proceed in order to solve the problem.

3 Self-similarity of the first kind

We now describe a physical process that, under appropriate assumptions and approximations, is described by a differential problem admitting a self-similar solution of the first kind.

We consider a two-dimensional gravity current (GC) of a homogeneous non-Newtonian power-law fluid flowing in a porous medium already saturated by a lighter fluid, with an impermeable horizontal bottom (Di Federico *et al.*, 2012), see figure 1. Under several hypotheses, the continuity equation becomes a non-linear PDE governing the local depth of the current, which can be written in dimensionless form as

$$\frac{\partial H}{\partial T} - \frac{\partial}{\partial X}\left[H\left|\frac{\partial H}{\partial X}\right|^{1/n-1}\frac{\partial H}{\partial X}\right] = 0, \tag{17}$$

where H is the depth of the current, X is the space, T is the time and n is the fluid behaviour index of the power-law fluid ($n < 1$ is shear-thinning, $n > 1$ is shear-thickening, $n = 1$ is Newtonian). The volume of the denser fluid is expressed as a power-function of the time:

$$\int_0^{X_N} H \mathrm{d}X = T^\delta, \tag{18}$$

where X_N is the front position and $\delta \geq 0$ is the exponent of the inflow rate, with $\delta = 0$ representing a constant volume and $\delta = 1$ a constant inflow rate. The boundary condition is $H(X_N) = 0$.

We already know (and this is a pragmatic approach) that self-similarity in most cases requires a group of scalings, hence we look for a group of transformations of the following type:

$$\begin{cases} X^* = \alpha X, \\ T^* = \beta T, \\ H^* = \gamma H. \end{cases} \tag{19}$$

Furthermore, we also know that the parameters α, β, γ are not functions of the variables, either independent or dependent. Finally, we expect a functional dependence between the three parameters that could lead to an infinity of solutions, since we can write two equations in three unknowns; the two equations are the mass conservation in the differential (17) and the integral (18) formulations, respectively.

All this information enables us to skip the analysis that allows (i) to ascertain that a self-similar solution is possible, and (ii) to identify the group of transformations.

At this point, in order to check the existence of the group, we substitute the expressions in (19) into the two equations (17-18) and then we impose invariance, i.e.:

$$\begin{aligned} &\frac{\partial H^*}{\partial T^*} - \frac{\partial}{\partial X^*}\left[H^* \left|\frac{\partial H^*}{\partial X^*}\right|^{1/n-1} \frac{\partial H^*}{\partial X^*}\right] \equiv \\ &\qquad \frac{\partial H}{\partial T} - \frac{\partial}{\partial X}\left[H \left|\frac{\partial H}{\partial X}\right|^{1/n-1} \frac{\partial H}{\partial X}\right] = 0 \rightarrow \\ &\qquad \frac{\beta}{\gamma}\frac{\partial H}{\partial T} - \frac{\alpha^{1/n+1}}{\gamma^{1/n+1}}\frac{\partial}{\partial X}\left[H \left|\frac{\partial H}{\partial X}\right|^{1/n-1} \frac{\partial H}{\partial X}\right] = \\ &\qquad\qquad \frac{\partial H}{\partial T} - \frac{\partial}{\partial X}\left[H \left|\frac{\partial H}{\partial X}\right|^{1/n-1} \frac{\partial H}{\partial X}\right], \end{aligned} \tag{20}$$

and

$$\begin{aligned} &\int_0^{X_N^*} H^* \mathrm{d}X^* - (T^*)^\delta \equiv \int_0^{X_N} H \mathrm{d}X - T^\delta = 0 \rightarrow \\ &\qquad \frac{1}{\alpha\gamma}\int_0^{X_N} H \mathrm{d}X - \frac{1}{\beta^\delta} T^\delta = \int_0^{X_N} H \mathrm{d}X - T^\delta, \end{aligned} \tag{21}$$

equivalent to

$$\begin{cases} \dfrac{\beta}{\gamma} = \dfrac{\alpha^{1/n+1}}{\gamma^{1/n+1}}, \\ \alpha\gamma = \beta^{\delta}. \end{cases} \tag{22}$$

The solution of the system (22) in terms of α is

$$\begin{cases} \beta = \alpha^{F_1}, \\ \gamma = \alpha^{F_2}, \\ F_1 = \dfrac{n+2}{n+\delta}, \quad F_2 = \dfrac{(n+1)\delta - n}{n+\delta}, \end{cases} \tag{23}$$

hence we are dealing with a one-parameter scaling group of transformations leaving invariant the differential problem. In other terms, we have identified the three scaling ratios of the variables that allow complete similarity between the space (X, T, H) and the space (X^*, T^*, H^*).

So far, no real advantage. However, we can reason in the logic of Buckingham theorem: we select a variable as the independent one, e.g. time; the transformations allow the computation of the dimension of both X and H with respect to T. Preservation of the intrinsic value of variables requires that they are a power-functions of the fundamental variable (see Longo, 2022, §1.2.4), i.e. $X = T^r$ and $H = T^s$, where the exponents r and s are calculated so that X and H are invariant in scaling (19), i.e.:

$$\begin{cases} X - T^r \equiv X^* - (T^*)^r = 0 \to \\ \qquad X - T^r \equiv \alpha X - \alpha^{rF_1} T^r = 0, \\ \\ H - T^s \equiv H^* - (T^*)^s = 0 \to \\ \qquad H - T^s \equiv \alpha^{F_2} H - \alpha^{sF_1} T^s = 0, \end{cases} \tag{24}$$

resulting in

$$\begin{cases} r = \dfrac{1}{F_1} \equiv \dfrac{n+\delta}{n+2}, \\ \\ s = \dfrac{F_2}{F_1} \equiv \dfrac{(n+1)\delta - n}{n+2}. \end{cases} \tag{25}$$

The differential problem, described by the typical equation

$$f\left(X, T, H\right) = 0, \tag{26}$$

applying Buckingham theorem can now be expressed as

$$f_1\left(\frac{X}{T^{1/F_1}}, \frac{H}{T^{F_2/F_1}}\right) = 0, \tag{27}$$

where the arguments are both invariant in the group of scaling transformations (19), and where the time T cannot be present autonomously.

At this point we have no further data or information to identify the structure of the function f_1; for progress, we assume the following structure

$$\frac{H}{T^{F_2/F_1}} = a\phi\left(\frac{X}{T^{1/F_1}}\right), \tag{28}$$

or

$$H = aT^{F_2/F_1}\phi(\eta), \quad \eta = \frac{X}{T^{1/F_1}}, \tag{29}$$

where a is a coefficient introduced to achieve a relevant simplification and generalisation of the differential problem, as we shall see later.

By substituting (29) into the original differential problem (17–18), including the boundary condition, yields

$$a^{1/n}\left(\phi\left|\phi'\right|^{1/n-1}\phi'\right)' - \frac{1}{F_1}\eta\phi' + \frac{F_2}{F_1}\phi = 0, \quad \phi(\eta_N) = 0, \tag{30}$$

and

$$a\int_0^{\eta_N}\phi\,\mathrm{d}\eta = 1, \tag{31}$$

where η_N is the value of η at the front of the current.

It is now convenient to normalize the variable η as $\chi = \eta/\eta_N$, obtaining

$$\frac{a^{1/n}}{\eta_N^{1/n+1}}\left(\phi\left|\phi'\right|^{1/n-1}\phi'\right)' - \frac{1}{F_1}\chi\phi' + \frac{F_2}{F_1}\phi = 0, \quad \phi(1) = 0, \tag{32}$$

and

$$a\eta_N\int_0^1\phi\,\mathrm{d}\chi = 1. \tag{33}$$

By imposing that the coefficient of the first term in (32), $a^{1/n}/\eta^{1/n+1}$, equals unity, results in $a = \eta_N^{n+1}$, with:

$$\left(\phi\left|\phi'\right|^{1/n-1}\phi'\right)' - \frac{1}{F_1}\chi\phi' + \frac{F_2}{F_1}\phi = 0, \quad \phi(1) = 0, \tag{34}$$

By substituting into eq.(33) results

$$\eta_N = \left(\int_0^1\phi\,\mathrm{d}\chi\right)^{-1/(n+2)}. \tag{35}$$

With these substitutions, we have obtained a differential problem where the only parameters are n and δ, with the integral part decoupled from the differential part: the integral can be calculated once and for all after solving the differential problem. The differential problem in the variable $\phi(\chi)$ can be integrated according to the scheme detailed in Di Federico *et al.* (2012).

Note that (i) self-similarity is strongly conditioned by the structure of the differential problem; in particular, if the integral condition of mass conservation (18) does not include a change in the volume of time as a power-function, self-similarity is lost in the most general case and it is necessary to numerically integrate the differential problem; (ii) some working techniques, such as the normalisation of the variable and the introduction of a coefficient in the expression of the self-similar dependent variable, simplify the achievement of the result to a very great extent.

In other cases, it is possible to separate the early time solution from the late time solution, neglecting terms that are initially small in the former case, or become small in the latter case, obtaining two asymptotic solutions, possibly with a matching expansion in the midway.

The experimental verification of this model was performed for the more general case where there is a porosity/permeability gradient in the horizontal direction or in the vertical direction, in the Hele-Shaw analogy, see Ciriello *et al.* (2016).

In summary, having identified a one-parameter scaling transformation has enabled the reduction of the number of variables from three to two. It follows that a partial differential problem with one dependent and two independents variables, turns into a differential problem with an ODE, which is always easier to solve than a PDE; in lucky cases the ODE admits analytical solutions.

3.1 Changing system of units of measurements requires a scaling group of transformations

Considering that the transformation from a system of units of measurement to another system of units of measurement of the same class (characterised by the same fundamental quantities, e.g. mass, length and time) is a scaling with a number of parameters equal to the number of fundamental quantities, it is evident that Buckingham theorem is simply the natural consequence of a principle of covariance that, in the case of physical processes, does not need the demonstration of invariance: it is perfectly reasonable that the same physical process is intrinsically invariant when the system of units used to measure it varies (in other words, the physical process does not notice the observer or the way it is observed and measured), and it follows that the physical process can be expressed with variables that are independent of the selected system of units, i.e. in terms of dimensionless groups, namely.

From this point of view, given two systems of units belonging to the same class, i.e. sharing the same fundamental quantities (e.g. mass M, length L and time T), the preservation of the intrinsic value of the quantities requires that the two systems are related by the following scaling group of transformations, with as many parameters as there are fundamental quantities:

$$\begin{cases} M^* = r_M M, \\ L^* = r_L L, \\ T^* = r_T T, \end{cases} \tag{36}$$

where the parameters are r_M, r_L, r_T. In this respect, a dependent variable like, e.g., the volumetric flow rate Q, has a scaling function of r_M, r_L and r_T (or a subset), and indeed invariance requires that $Q^* = r_L^3 r_T^{-1} Q$ or $r_Q r_L^{-3} r_T = 1$.

The reduction in the number of variables is equal to the number of fundamental variables, i.e. of the number of parameters of the group of transformations.

4 Who is right and who is wrong?

The heading of this paragraph refers to the truthfulness or otherwise of the self-similar solution. In essence, the self-similar solution obtained is not the rigorous solution of the differential problem, but it can be an excellent approximation that works well neither too early nor too late (early or late refer to non-stationary problems in which the time variable appears). In this sense, the answer to the question is a paraphrase, or rather, an oxymoron, of a friend of mine: we could answer "You are right, but I am not wrong".

In the initial phase of the process, boundary conditions can be decisive and such that the model deviates from physical reality (e.g. the curvature of the trajectories may be so high that the assumption of hydrostatic pressure distribution, which underlies the shallow water equations, is incorrect), and the analytical scheme is conditioned to such an extent that self-similarity has not yet been established. In the terminal phase the self-similar solution fails once again either because other assumptions of the analytical model fail, or because of perturbations that may finally have the upper hand.

At this point the question arises: can it be the case that the initial and boundary conditions are always non-negligible with respect to the internal dynamics of the process? Yes, it can happen, and in such a case the self-similar solution, although predicted by theory, is not an acceptable approximation. A second question is: can it happen that perturbations immediately "pollute" the flow field? Again, the answer is yes, since the system may be so unstable that it amplifies the infinitesimal perturbations, possibly with successive bifurcations. It follows from these arguments that there is an absolute need for experimental validation to confirm or deny the adequacy of a self-similar solution, possibly indicating the limits of validity of the approximation (see Ball & Huppert, 2019; Ball *et al.*, 2017).

As an example, consider the case of the dipole, a GC propagation process where the first-order momentum of the fluid volume is conserved (Longo *et al.*, 2015). Figure 2 shows the evolution of a dipole for a non-Newtonian power-law fluid, experimentally reproduced in a Hele-Shaw cell. The fluid is initially stored in a finite portion of the Hele-Shaw cell (the physical approximation of

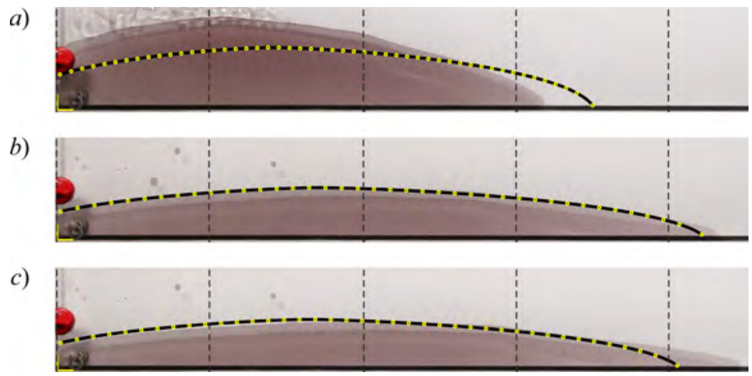

Figure 2: Time evolution of a dipole for power-law GCs. *a*) Early time, *b*) intermediate asymptotic, and *c*) late time. Curves and bullets are theory. (modified from Longo *et al.*, 2015)

a Dirac delta), bounded by two vertical sluice gates; at a later stage, after the sluice gates are lifted, the fluid recedes to the left falling out of the domain, and propagates as a GC to the right. While at the beginning the deviation between theory and experiment is visible, later on an adequate agreement is reached, which, in a third phase, is lost again. At the beginning, the deviation is determined by the fact that (i) a finite volume of fluid cannot be completely representative of a Dirac delta, and (ii) the manoeuvre of opening the sluice gates, one on the right and the other on the left, occurs in a finite time and with asymmetries. In the terminal phase, the deviation results from the fact that the surface tension is no longer negligible and, furthermore, the dynamics of the flow is controlled by both the shear in the horizontal plane and the shear in the vertical plane (the simplified model only includes the first of the two).

Strictly speaking, we should compare the self-similar solution with the numerical solution, so as not to involve the conceptual model adopted to schematise the physical process: the conceptual model, in fact, could be fallacious and experiments could mercilessly highlight these limitations. This does not mean that the self-similar solution that cannot reproduce the experiments would be incorrect. Thus the direct comparison of the self-similar solution with experiments, if satisfactory, gives good reasons for the acceptability of the numerous approximations used both in the conceptual model and in the set-up of self-similarity.

5 Self-similarity of the second kind

Until now, we have extended to the study of differential problems some concepts that, more or less unconsciously, we already applied in dimensional analysis, first and foremost Buckingham theorem. Some distinction of method is in order: when we deal with dimensions of physical quantities, dimensionless groups are derived by combining the variables involved so that they have zero dimension, precisely; when instead we have calculated self-similar variables, power-functions of variables already rendered dimensionless by choice, we have imposed invariance with respect to scalings which leave the differential problem unchanged.

In both cases the dimensional calculus and the simple application of the covariance principle are sufficient to obtain invariant groups, in the first case with respect to different unit systems, in the second case with respect to the Lie group. We remind that the selection of dimensionless groups is not unique, in the sense that the combinations of variables that determine a dimensionless group are infinite, and Buckingham theorem only establishes the maximum number of dimensionless groups sufficient to describe the physical process, but does not in any way indicate what these groups are; similarly, the number of scaling parameters allows the number of variables sufficient to describe the differential problem to be reduced by the same value, but gives no indication on the selection of self-similar variables.

There are physical problems that, when translated in terms of differential problems, have a self-similar solution that is not detectable on the basis of the criteria and methods we used in § 3. In many cases, the discontinuity of a parameter, or the transition from an idealised problem to a problem that is more down-to-earth in physical reality, is enough to lead to the occurrence of what is termed *anomalous scaling*. The literature is quite extensive on this subject, with minute details on the occurrence of such problems (see Barenblatt, 1996, 2003).

In the anomalous scaling there appears at least one variable, a power-function of the starting variables, in which an exponent cannot be calculated a priori on the basis of the transformation that leaves the differential problem invariant. In some cases, the conservation of some property in the physical process intervenes to determine the value of that exponent, which indicates that a problem with an apparent anomalous scaling can be traced back to a problem with a self-similar solution of the first kind; in other cases, however, the exponent is derived by solving the problem, and is therefore referred to as *eigenvalue*.

Such an eigenvalue can be single, or belongs to a discrete group of finite or infinite numerosity, or can be a continuous function and thus be representable as a spectrum. If there are multiple eigenvalues, the problem arises as to which eigenvalue is the correct one, and this is where experimental practice comes in; based on the above comments, experiments are required to validate even the case of a single eigenvalue.

5.1 Converging gravity current modelled as a second kind self-similar problem

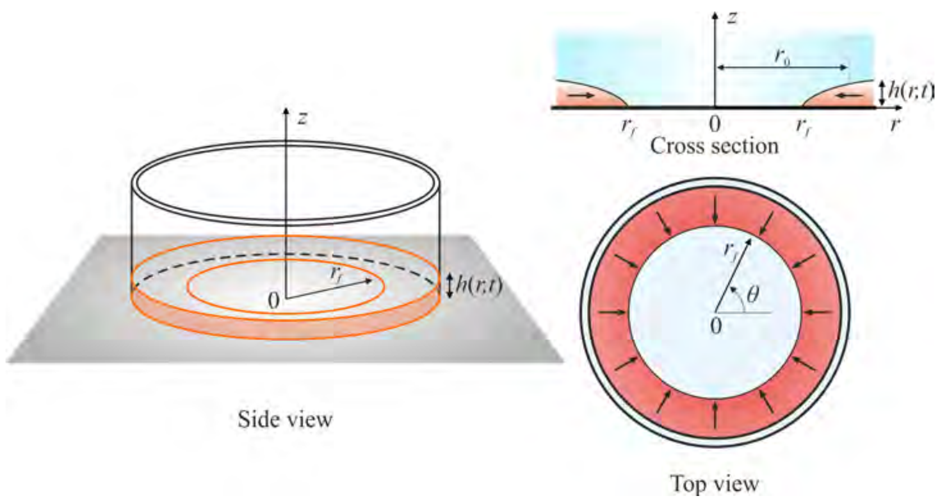

Figure 3: Radial converging flow. The model refers to $r_f \ll r_0$ and $r_f \ll h_0$. (modified from Longo *et al.*, 2021)

A classic problem representative of a second-kind self similar solution problem and that has received particular attention is the study of converging GCs, in cylindrical symmetry (also with experimental verification) and in spherical symmetry (only theoretical analysis) (Gratton & Minotti, 1990; Zheng *et al.*, 2014; Longo *et al.*, 2021). In cylindrical symmetry, we have a fluid that initially occupies only a portion of the domain in the shape of a circular crown, with a dry area towards the axis; then a circular cylindrical sluice gate is lifted and the fluid, under the action of gravity, advances towards the axis of symmetry (a dam-break), with a radial flow field with converging trajectories, see the schematic in figure 3.

First of all, we wonder what is the origin of the self-similarity of the second type. If we analyse what happens when the front has a small radius with respect to any other scale length that characterises the flow field (e.g., the initial radius and the depth of the fluid in the tank before the dam-break), then the parameters that could be influential in the evolution of the current lose their meaning. Hence, dimensional analysis as extended with groups theory cannot help in finding the self-similar variables.

Then, we ask what method should be adopted to identify the structure of the self-similar variables, if any, as well as to identify the additional parameter(s) characterising these variables.

The methods adopted are (i) numerical integration to find an asymptotic trend in the solution, which must appear linear on a bi-logarithmic scale if the dependence of the two represented variables is a power-function; (ii) the direct approach using, for example, a phase-plane analysis. In this second methodology, the procedure is a trial and error and the unknown exponent of the self-similar variable is called the eigenvalue since it originates from the structure of the differential problem describing the process.

The details of the analysis are reported in Longo *et al.* (2021) and are here only briefly recalled.

In a radial geometry flow field, neglecting the curvature of the trajectories we assume the shear stress acting on the plane of normal z in the radial direction r, τ_{zr}, to be dominant, and calculate a vertically averaged velocity of the non-Newtonian power-law fluid equal to

$$u(r,t) = -\mathrm{sgn}\left(\frac{\partial h}{\partial r}\right) h^{(n+1)/n} \frac{n}{2n+1} \left(\frac{\Delta\rho\, g}{\mu_0}\right)^{1/n} \left|\frac{\partial h}{\partial r}\right|^{1/n}, \tag{37}$$

where n is the fluid behaviour index, μ_0 is the consistency index, $\Delta\rho$ is the density difference between the fluid of the current and the ambient fluid, g is acceleration of gravity.

Mass conservation reads

$$\frac{\partial h}{\partial t} + \frac{1}{r}\frac{\partial (ruh)}{\partial r} = 0, \tag{38}$$

with the boundary condition $h(r_f) = 0$, where r_f is the front of the current; integral mass conservation reads

$$2\pi \int_{r_f}^{r_0} r h \,\mathrm{d}r = Q t^{\alpha}, \tag{39}$$

where $\alpha \geq 0$ is an exponent with $\alpha = 0$ corresponding to dam-break and $\alpha = 1$ corresponding to constant inflow rate; other values of α generate waxing and waning inflow rate. $Q > 0$ with $[Q] = L^3 T^{-\alpha}$ is the coefficient of the inflow rate.

We adopt the initial radius r_0 as horizontal length scale and balancing the terms in eqs.(37–39) results in a vertical length scale

$$h^* = \left(\frac{2n+1}{n}\right)^{\alpha n/(2\alpha+\alpha n+n)} \left(\frac{Q}{2\pi}\right)^{n/(2\alpha+\alpha n+n)} \times r_0^{[\alpha+(\alpha-2)n]/(2\alpha+\alpha n+n)} \left(\frac{\mu_0}{\Delta\rho\, g}\right)^{\alpha/(2\alpha+\alpha n+n)}, \tag{40}$$

a velocity scale

$$u^* = \frac{n}{2n+1} h^{*(n+2)/n} \left(\frac{\Delta\rho\, g}{\mu_0}\right)^{1/n} r_0^{-1/n}, \tag{41}$$

and a time scale

$$t^* = \frac{2n+1}{n} h^{*-(n+2)/n} r_0^{(n+1)/n} \left(\frac{\Delta\rho\, g}{\mu_0}\right)^{-1/n}. \tag{42}$$

In terms of the dimensionless variables $R = r/r_0$, $H = h/h^*$, $T = t/t^*$, and after introducing (37) in (38), the differential problem becomes:

$$\frac{\partial H}{\partial T} - \frac{1}{R}\frac{\partial}{\partial R}\left[RH^{(2n+1)/n}\left|\frac{\partial H}{\partial R}\right|^{1/n-1}\frac{\partial H}{\partial R}\right] = 0, \quad H(R_f) = 0, \tag{43}$$

$$\int_{R_f}^{1} RH\,\mathrm{d}R = T^{\alpha}. \tag{44}$$

Following the same approach adopted in § 3, the one-parameter transformations group leaving invariant the differential problem (43–44), is

$$\begin{cases} R^* = \beta R, \\ T^* = \beta^{F_1} T, \\ H^* = \beta^{F_2} H, \quad \text{where} \\ F_1 = \dfrac{3n+5}{2\alpha + \alpha n + n}, F_2 = \dfrac{\alpha + (\alpha-2)n}{2\alpha + \alpha n + n}, \end{cases} \tag{45}$$

and the self-similar variables, invariant within the same transformations group, are

$$H = aT^{F_2/F_1}\Psi(\xi), \quad \xi = RT^{-1/F_1}, \tag{46}$$

where a is a coefficient to be computed in order to simplify the results. Substituting these variables in (43–44), results in:

$$\left(\xi\Psi^{(2n+1)/n}\left|\Psi'\right|^{1/n-1}\Psi'\right)' + \frac{\xi^2}{F_1}\Psi' - \frac{F_2}{F_1}\xi\Psi = 0, \quad \Psi(\xi_f) = 0, \tag{47}$$

$$\int_{\xi_f}^{\xi_0} \xi\Psi\,\mathrm{d}\xi = 1, \tag{48}$$

where the prime indicates the derivative with respect to the argument.

The next step consists in normalising the variable ξ, but the difficulty arises immediately. In fact, there are two variables to be normalised in the integral condition (48) in order to allow their calculation, i.e. ξ_0 and ξ_f, but we only have one equation. Even the introduction of a stretching like $\chi = (\xi - \xi_0)/(\xi_f - \xi_0)$, mapping $[\xi_0, \xi_f] \to [0, 1]$ is not useful, since it introduces additional terms in (47–48) which cannot be matched with the other terms.

In principle, an asymptotic solution is possible when one of the two variables, e.g. ξ_0, is "forgotten" and is no longer relevant to the propagation of the current: the current evolves on the basis of local scaling quantities implicit in the differential problem, i.e. controlled by eigenvalues. In other words, we can also state that the presence of two length scales, i.e. r_0 and h^*, prevents self-similarity at least of the first kind, and indeed the presence of multiple scales is one of the signal of a missing self-similarity (see Barenblatt, 1996), although there are cases where the multiple scales collapse to the same dependence structure and allows self-similar solutions; see Di Federico *et al.* (2017) where, for a linearly waxing inflow rate a GC of Herschel-Bulkley fluid in a fracture and in a porous medium, admits self-similarity. See also Longo & Di Federico (2014).

We are therefore dealing with a differential problem that could admit a self-similar solution, but for which the criteria and methodologies used in § 3 are not applicable.

We now try a different approach.

The dependent variables u and h are expressed with dimensionless counterpart U and H as below:

$$u(r,t) = \frac{r}{t_r} U(r,t_r), \tag{49a}$$

$$h(r,t) = \left(\frac{2n+1}{n}\right)^{n/(n+2)} \left(\frac{\mu_0}{\Delta\rho\, g}\right)^{1/(n+2)} \frac{r^{(n+1)/(n+2)}}{t_r |t_r|^{-2/(n+2)}} H(r,t_r), \tag{49b}$$

where $t_r = t - t_c$ and with t_c the touch-down time required for the current to reach the central axis $r = 0$. The absolute value in (49b) is introduced to allow t_r to be negative during current propagation toward the origin (filling or pre-closure phase), and positive after the front of the current has reached the origin (levelling or post-closure phase); the second phase of evolution also admits a self-similar solution. Note that the dimensionless variables adopt "local" scales, i.e. (i) the contact time, whilst the current release time (a "far" variable) scaled as r_0/u^* is no longer relevant; (ii) the distance r from the origin, which is not affected by r_0, if r_0 is large enough. The new velocity scale r/t_r looks anomalous compared to traditional scales, which are generally a function of parameters and not of the variables involved in the physical process. In fact, this very definition will lead to an eigensolution.

Substituting eqs.(49a–49b) into eqs.(37–38), yields:

$$(n+2)H\,|H|^n\, r\frac{\partial H}{\partial r} + (n+1)H\,|H|^{n+1} + (n+2)U\,|U|^{n-1} = 0, \tag{50a}$$

$$-(n+2)r\frac{\partial HU}{\partial r} + (n+2)t_r\frac{\partial H}{\partial t_r} - (3n+5)UH - nH = 0. \tag{50b}$$

We aim to find a group of transformations that leave invariant eqs.(50a–50b):

$$\begin{cases} U^* = \alpha U, \\ H^* = \beta H, \\ r^* = \gamma r, \\ t_r^* = \omega t_r; \end{cases} \tag{51}$$

substituting into eqs.(50a–50b), yields

$$\beta^{n+2} = \alpha^n, \quad \alpha\beta = \beta, \tag{52}$$

which admits only the trivial solution $\alpha = \beta = 1$ and leaves undetermined the other two coefficients γ and ω, so there is no useful transformation.

However, in the same frame of mind of looking for process scales as local scales contained in the differential problem, we look for the parameters of the group of transformations as

$$\begin{cases} U^* = U, \\ H^* = H, \\ r^* = \gamma r, \\ t_r^* = \gamma^{1/\delta} t_r, \end{cases} \tag{53}$$

where δ is an unknown exponent. Three possible variables invariant within this group are

$$H, \quad U, \quad \frac{r}{t_r^\delta}, \tag{54}$$

hence the differential problem can be formally expressed as

$$f\left(H, U, x, t_r\right) = 0 \to g\left(H, U, \frac{r}{t_r^\delta}\right) = 0. \tag{55}$$

Eqs.(50a–50b) are rearranged to obtain

$$\frac{\mathrm{d}U}{\mathrm{d}H} = \frac{H|H|^{n+1}[2(n+2)U - (n+1)\delta + n] - (U+\delta)(n+2)U\,|U|^{n-1}}{H[(n+1)H|H|^{n+1} + (n+2)U\,|U|^{n-1}]}, \tag{56a}$$

$$\frac{\mathrm{d}\ln\xi}{\mathrm{d}H} = -\frac{n+2}{(n+2)H^{-1}|H|^{-n}U\,|U|^{n-1} + (n+1)H}, \tag{56b}$$

where $\xi = r t_r^{-1}|t_r|^{1-\delta}$ is the self-similar independent variable which has been embedded in its logarithm, with $\mathrm{d}\,\xi/\xi \equiv \mathrm{d}(\ln\xi)$. Note that the variable ξ is dimensional, with $[\xi] = LT^{-\delta}$; however, if it is involved in the analysis by its logarithm, the new variable $\ln\xi$ is dimensionless, since $[\ln\xi] \equiv [\mathrm{d}\,\xi/\xi] = 0$. These two equations are a set of autonomous planar ODEs, with boundary conditions represented by points in the phase space. We define as *singular points* the simultaneous zeros of numerator and denominator of eq.(56a), with a further singular point obtained by setting the denominator of eq.(56a) to infinity. Hence, there are four singular points, namely

$$\begin{cases} \mathrm{O}: (H,U) \equiv (0,0), \\ \mathrm{A}: (H,U) \equiv (0,-\delta), \\ \mathrm{B}: (H,U) \equiv \left(\left[\dfrac{n+2}{n+1}\right]^{1/(n+2)} \left[\dfrac{n}{5+3n}\right]^{n/(n+2)}, -\dfrac{n}{5+3n}\right), \\ \mathrm{C}: (H,U) \equiv \left(-\infty, \dfrac{(n+1)\delta - n}{2(n+2)}\right). \end{cases} \tag{57}$$

A possible solution to the differential problem is represented by a curve, in phase space, joining the two critical points O and A in the filling phase, joining O and C during levelling (see Sachdev, 2019; Zheng *et al.*, 2021). The integration of the differential problem is performed, for example, by setting a value of δ of first attempt and starting the integration from either A or O. In general, the two branches do not meet (see figure 4a) and a different value of δ must be selected, proceeding until the contact between the two curves is obtained (see figure 4b). The value of δ that results in the contact between the two curves is the *critical eigenvalue*, δ_c.

Figure 4a shows the phase portraits for $\delta \neq \delta_c$ and figure 4b shows the phase portraits for $\delta = \delta_c$, also including the behaviour in the levelling phase, joining O with the asymptote C.

At this point, it is straightforward to calculate the front position:

$$\frac{r_f}{r_0} = k \left(\frac{t_c - t}{t_c} \right)^{\delta_c}, \tag{58}$$

where k is a coefficient to be evaluated through numerical integration or experiments.

As highlighted from the very beginning of the present article, the soundness of the analysis and the correctness of the results require experimental validation. Figure 5 shows the front position in time for experiments with a Newtonian fluid. Note that the adaptation of the theory to the experiments takes place after a certain time interval from the release of the current, as embedded in the self-similarity model. The late time behaviour of the front position deviates again from the model, and in this sense the concept of intermediate asymptotic holds also for the second kind self-similarity: "not too early, not too late".

Similar data are also available for the levelling phase (not shown), when the depth of the current in the origin varies in time following a power-function obtained as an expansion of eq.(56b) about C.

6 Conclusion

The adoption of problem-solving techniques searching for self-similar solutions has advanced enormously from its origins, in the latter part of 19th century, to the present day. In fact, a great deal of agility of thought is required to commensurate physical rigour and mathematical approximation, as well as to make the problem solvable analytically or numerically.

The self-similar solutions of the first kind appear to be more readily understood, since they are framed within dimensional analysis and dimensionless groups referred to by Buckingham theorem. Self-similar solutions of the second kind appear more intriguing, with a group of transformations itself unknown and obtained from the solution of the differential problem.

The perspective designed by Lie, and applied by numerous scientists in the most diverse fields of physics, allows for both the advancement towards the solution of new problems and the reinterpretation of classical problems. In this regard, we consider that a sea gravity wave of infinitesimal amplitude is described by a periodic function of argument $x - ct$, where x and t are space and time, respectively, and c is the phase celerity, which is computed from the

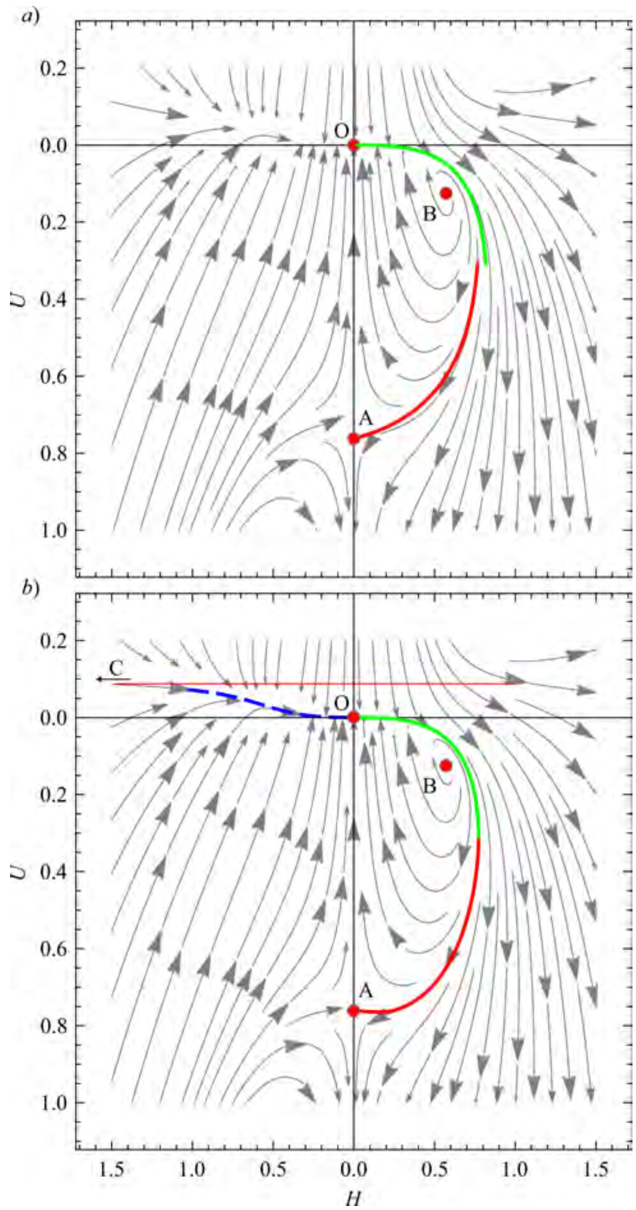

Figure 4: Converging radial GC. a) Phase portrait of (56a) for $n = 1$ (Newtonian fluid) with a first attempt value of δ, and b) with $\delta = \delta_c = 0.7620351$. The continuous curve refers to the pre-closure phase, the dashed curve refers to the post-closure (levelling) phase, the thin red horizontal line indicates the asymptote in the levelling phase.

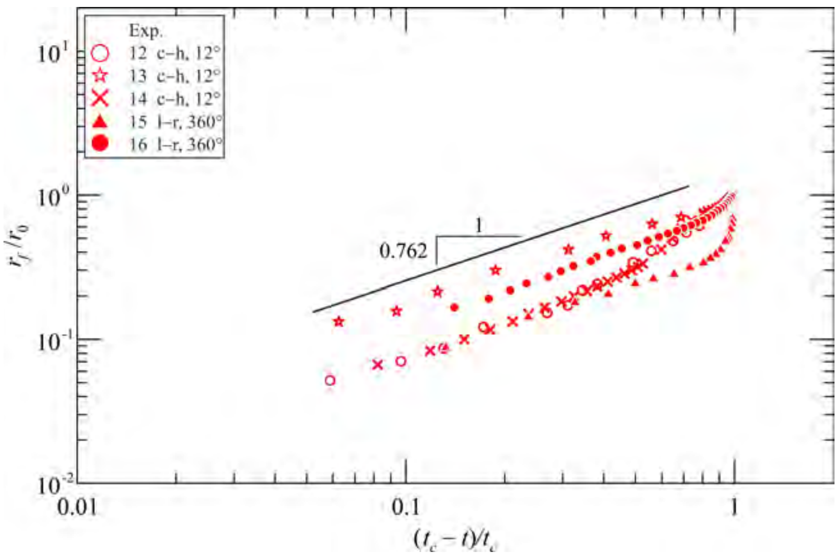

Figure 5: Front position of GCs in radial converging flow, Newtonian fluids. Symbols refer to experiments, the straight line corresponds to the theoretical curve for $n = 1$ with eigenvalue $\delta_c = 0.762$. (modified from Longo *et al.*, 2021)

dispersion equation. In a self-similarity frame, the sea gravity wave is described by a function of the self-similar variable ξ/τ^c where $x = \ln \xi$ and $t = \ln \tau$, and where the celerity is the exponent.

In all these analyses, experimental validation is also of primary importance, without which nothing can be said about the feasibility of the physical process following the results: we are not fooled by mathematical stability or by the reasonableness of the results, we believe in experimental evidence.

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