FORCHHEIMER POROUS GRAVITY CURRENTS

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KEY POINTS

- *A mathematical model is presented for predicting the final shape of a gravity current under Forchheimer regime.*
- *The problem is approximated for different values of local Forchheimer number, deriving self-similar solutions.*
- *Numerical and experimental results are used to successfully validate the approximations*

1 INTRODUCTION

Gravity currents (GCs), generated by the injection of a dense fluid into a light ambient fluid, or vice versa, occur in several natural phenomena and industrial processes (Huppert 2006). Conventionally, to build the mathematical model describing such phenomena in porous medium, the Darcy regime is assumed to be dominant i.e., slow, laminar, and viscous flow in uniform porous medium under steady-state condition. Thus, it is considered that the propagation of the GCs is governed by the balance between viscous and buoyancy forces and the inertial ones are being neglected. However, in several groundwater contamination scenarios, fluids spreading into porous systems exhibit inertial behavior. The onset of non-Darcy flow or Forchheimer flow is not well determined (Zeng and Grigg 2006) and there are many speculations around the causes of such transition. But as a general observation, increases in the seepage velocity leads to gradual deviation from Darcy regime, in which the relation between velocity and the deriving head/pressure gradient is no longer linear (Arthur 2018). This deviation is introduced by adding an additional term, quadratic in the velocity, to the Darcy law formulation.

Under the Darcy regime, velocity of a GCs is calculated by integrating the momentum balance between pressure and shear forces. This velocity then used in the local mass conservation to find the current profile. The resulting equation is a non-linear PDE which cannot be solved by conventional analytical methods thus scaling and similar solution techniques are applied along with numerical methods (see Barenblatt 1952).

In this study, a release of fluid over an impermeable substrate (plane geometry) is considered to be Forchheimer flow. The homogeneous isotropic medium is saturated and surface tension and viscous fingering effect are neglected so that a sharp interface exists between the ambient and intruded fluids. The vertical velocity is taken to be negligible, hence the hydrostatic condition and the lubrication approximation hold (see figure1). Forchheimer velocity in the porous medium is calculated after Auriault et. al. (2007) and approximations are made, based on local Forchheimer number (ζ), to simplify the equation. After eliminating dimensions, the flow law is inserted in the local mass conservation to derive the mathematical model. Similarity variables are then introduced in order to find shape functions and asymptotic solutions. Also, a full numerical solution to the non-linear PDE is provided with given initial and boundary conditions. This solution is then compared with the approximate self-similar solutions for verifying the approximations based on the low and high local Forchheimer number. These solutions are calculated for different types of injection rate.

2 THEORY

The vectorial form of Darcy-Forchheimer equation in an isotropic medium read as:

$$\nabla p - \rho \mathbf{g} = -\left(\frac{\mu}{k} + \rho\beta|\mathbf{u}|\right)\mathbf{u} = -(\alpha + \rho\beta|\mathbf{u}|)\mathbf{u} \tag{1a}$$

where p is the pressure, ρ is the density, $\mathbf{g} \equiv (0, 0, -g)$ is the gravity, $\mathbf{u}$ is the Darcy velocity of modulus $|\mathbf{u}| = \sqrt{\mathbf{u}\cdot\mathbf{u}}$, k ($[k]$ = [L^2]) is the medium permeability, α is the ratio of dynamic viscosity over intrinsic permeability (μ/k), and β ($[\beta]$ = [L^{-1}]) is the inertial factor, having an empirical nature. Equation (1a) yields:

$$|\mathbf{G}| = -\nabla(p + \rho g z) = (\alpha + \rho\beta|\mathbf{u}|)|\mathbf{u}|. \tag{2a}$$

Eq. (2a) is a second order algebraic equation, hence (see Auriault et. al. (2007))

$$|\mathbf{u}| = \frac{\sqrt{\alpha^2 + 4\beta\rho|\mathbf{G}|} - \alpha}{2\rho\beta}. \tag{3a}$$

Substituting eq (3a) in (1a) gives after some algebra for the one-dimensional case, i.e. the GC problem depicted in Figure 1

$$u = -sgn(G)\frac{\alpha}{2\rho\beta}\left(\sqrt{1 + \frac{4\rho\beta|G|}{\alpha^2}} - 1\right), \tag{4a}$$

where $G = -\partial p/\partial x$, which is equal to $(\Delta\rho g\ \partial h/\partial x)$ under the lubrication approximation (see Huppert & Woods 1995). Thus, the velocity in the Forchheimer regime for 1-D free-surface flow in porous media reads

$$u = -sgn\left(\frac{\partial h}{\partial x}\right)\frac{\alpha}{2\rho\beta}\left(-1 + \sqrt{1 + 4N\left|\frac{\partial h}{\partial x}\right|}\right), \tag{5a}$$

$$N = \frac{\Delta\rho\beta\rho g}{\alpha^2}, \tag{5b}$$

where $\Delta\rho$ is the density variation between the intruding and ambient fluid. Inserting the dimensionless form of Eq. (5a) into local mass conservation along with global mass conservation and its boundary condition, gives the final form of the problem, where capital letters indicate dimensionless coordinates:

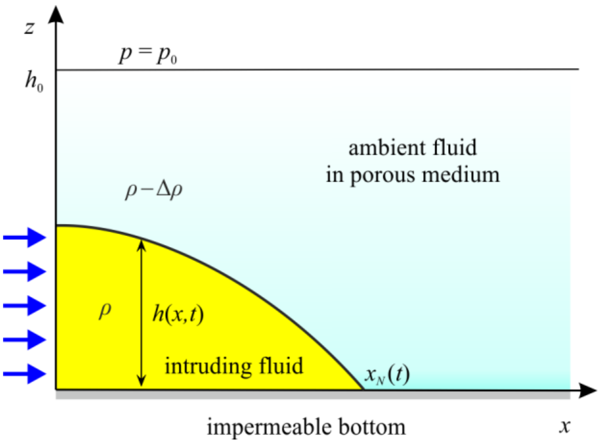

Figure 1. Schematic of the problem.

$$\frac{\partial H}{\partial T} = \frac{1}{2}sgn\left(\frac{\partial H}{\partial X}\right)\frac{\partial}{\partial X}\left[H\left(-1 + \sqrt{1 + 4N\left|\frac{\partial H}{\partial X}\right|}\right)\right], \tag{6a}$$

$$\int_0^{X_N} H\, dX = T^\gamma, \quad H(X_N(T), T) = 0. \tag{6b}$$

Note that here γ is the injection rate, for which $\gamma = 0$ indicates a constant volume and $\gamma = 1$ a constant injection. $\gamma = 2$ is a special case that needs different scales for the dimensionless form.

Equation (6a) is a non-linear PDE and cannot be solved with conventional methods. Thus, by defining the local Forchheimer number as $(\zeta = 4N|\partial H/\partial X|)$ and approximating the exact equation (6a) based on the assumptions $\zeta \ll 1$ and $\zeta \gg 1$, the problem is reduced to a classical form amenable to a self-similar solution.

The value of ζ depends both on the N value, which is fixed for specific problem and the local slope of gravity current. The following sections provide the solutions for $\gamma \neq 2$ based on low local Forchheimer number.

3 LOW LOCAL FORCHHEIMER NUMBER $(\zeta \ll 1)$

Consider small values of ζ, but not so small as to make inertial effects negligible. So, the quantity $\sqrt{1+\zeta}$ is approximated by its second-order Taylor's expansion $1 + \zeta/2 + \zeta^2/8$ .Using this approximation yields the classical form of the porous medium equation self- similar GCs:

$$\frac{\partial H}{\partial T} = N \frac{\partial}{\partial X}\left[H \frac{\partial H}{\partial X}\right] \tag{7}$$

By scaling and introducing similarity variables, the nonlinear PDE (7) reduces to a nonlinear ODE. A similarity variable is defined as $\xi = XT^{-(\gamma+1)/3}$ and the similarity scaling is defined as $H\,(X,T) = \xi_N^2 T^{2\gamma-1/3}\Phi(\chi)$, where ξ_N is the unknown value of ξ at the current front, $\Phi(\chi)$ is the shape function or thickness profile, and $\chi = \xi/\xi_N$ is a rescaled similarity variable, see the full methodology outlined in Di Federico et al. (2012). After quite some algebra, the final ODE reads:

$$N(\Phi\Phi')' + \frac{\gamma-1}{3}\chi\Phi' - \frac{2\gamma-1}{3}\Phi = 0, \tag{8a}$$

$$\xi_N = \left(\int_0^1 \Phi\, d\chi\right)^{-1/3}, \Phi\,(1) = 0, \tag{8b}$$

where primed terms indicate the ordinary derivative with respect to χ. For $\gamma = 0$, the problem has a closed-form analytical solution as:

$$\Phi(\chi) = \frac{1}{6N}(1-\chi^2) \quad , \quad \xi_N = (9N)^{1/3}\,. \tag{9}$$

For $N = 1$, eq. (9) again reduces to the result by Huppert & Woods (1995) and Pattle (1959).

Numerical methods must be employed to solve eqs. (8a-8b). A second boundary condition near the toe of the current is introduced to this purpose by expanding eq.(8a) in a Frobenius series for $\chi \to 1-$, finding:

$$\Phi = \frac{\gamma+1}{3}\,(1-\chi), \tag{10a}$$

$$\Phi'(\chi-1) = -\frac{\gamma+1}{3}. \tag{10b}$$

Figure 2 shows the results of the integration of eq. (8a) and eq. (9), with different values for N and γ. Note that for $\gamma < 2$ the average slope of the current decays with time, which means that the model of a thin long current is correct and that whatever the value of N, the parameter ζ becomes asymptotically small, guaranteeing that the solution of the differential problem represented by eqs.(8a-8b) is the correct one in the long run. A different analytical treatment is needed for the case of high local Forchheimer number ($\zeta >> 1$), leading to a modified version of Figure 2.

4 THE EXPERIMENTS

To validate the model, we have started a series of experiments in a channel filled with glass beads with diameter d = 4 mm. Fluid is injected with a vane pump controlled with an inverter through a Labview software, able to reproduce time variable inflow rate. The front position and the current profile are detected with video image analysis. Figure 3a shows three snapshots of an experiment with constant inflow rate. A profile with down convexity is expected. Figure 3b shows the time exponent of the front position. After forgetting the initial conditions, for x > 50 cm, the GC advances initially in the high local Forchheimer number regime, with α = 3/4 for 50 < x < 100 cm, and then evolves toward the low local Forchheimer number regime with α = 2/3.

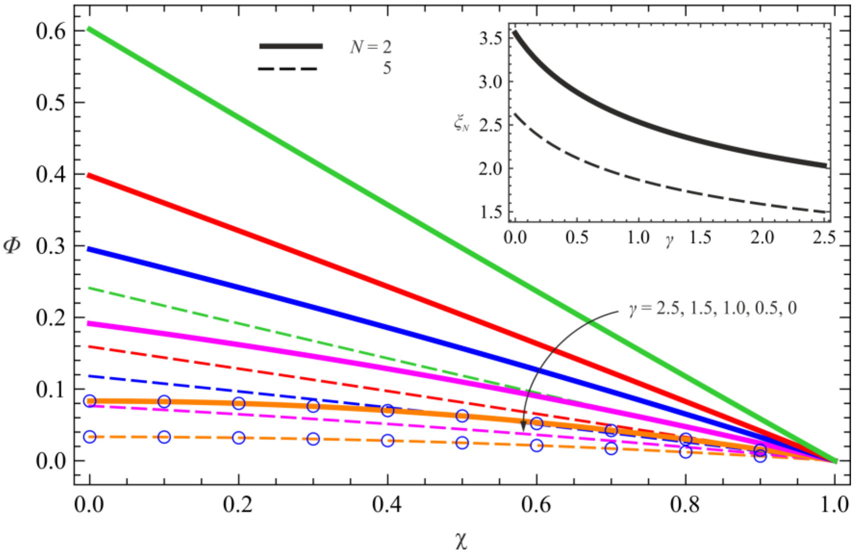

Figure 2. Self-similar solution for low local Forchheimer number, first order approximation. Shape function Φ as a function of scaled similarity variable χ for different values of γ and for N = 2, 5. Curves represent numerical solutions of (8a) for γ ranging between 0 and 2.5; open circles represent the analytical solution (9) for γ = 0. The inset shows the value of the prefactor ξ_N.

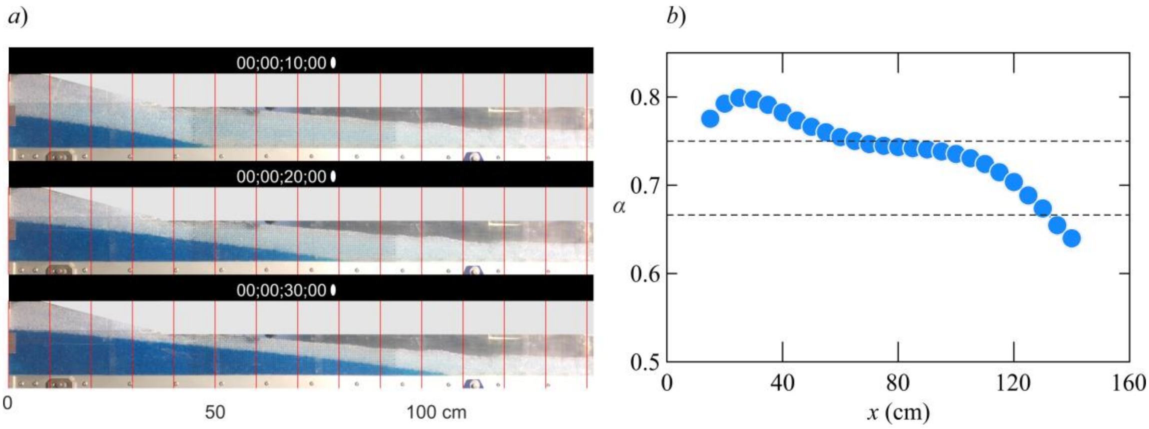

Figure 3. *a*) Three snapshots of an experiment with constant inflow rate 80 ml/s. *b*) Experimental time exponent of the front position. The dashed horizontal lines correspond to the high local Forchheimer number (3/4) and to the low local Forchheimer number (2/3), respectively. Water added with blue dye advancing in a porous medium of glass beads *d* = 4 mm.

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