HYDRAULIC JUMP ASSOCIATED WITH AN ABRUPT CHANNEL DEVIATION IN LAMINAR FLOW OF YIELD STRESS FLUIDS

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KEY POINTS

- *This study analyses the effect of a sudden deflection of the lateral channel wall on the laminar flow of a Herschel-Bulkley fluid, which can cause a hydraulic jump.*
- *The deflection originates a perturbation wave, and the phenomenon is described by a four equations system.*
- *From the general case of a Herschel-Bulkley fluid, the special cases for Bingham, Ostwald-DeWaele and Newtonian fluids are derived.*

1 INTRODUCTION

A hydraulic jump occurs when a free surface flow passes abruptly from supercritical to subcritical regime, i.e. from a lower depth Y_1 to a greater depth Y_2; upstream of the jump, much of the energy is kinetic due to the high velocity of the flow; downstream, the flow is slower and only a small portion of the energy is kinetic. The transition happens due to a change in the boundary conditions, such as a variation of the geometry of the channel, a change of roughness of the wall or a change of slope. Hydraulic jump in laminar flows of Newtonian and non-Newtonian fluids has been studied in the context of mine tailings flow in artificial channels (*Haldenwang & Slatter*, 2006; *Burger et al.*, 2010); some earlier contributions on the topic for rectilinear flow are *Zhou et al.* (2007) for Bingham fluids and *Ugarelli & Di Federico* (2007) for Herschel-Bulkley fluids.

In general, a free surface flow encountering a change in the geometry of the channel undergoes a perturbation of its characteristics; the entity of the perturbation varies greatly, depending on the upstream flow conditions, the nature of the geometric change and, for viscous flows, the rheological properties of the fluid. For inviscid flows, the case of an abrupt deviation of the channel axis, which in turn generates, under certain conditions, a hydraulic jump, has been studied by *Ippen* (1936) and *Ippen & Knapp* (1936). They also extended the analysis to a continuously curving channel (*Marchi & Rubatta*, 1981).

To our knowledge, there is a general lack of literature on the conditions for the generation of a hydraulic jump in laminar flows in an abruptly deflected channel. In the following, we adopt a Herschel-Bulkley model to adequately represent the rheologic behavior of mine tailings, slurries and industrial fluids; the effect of a hydraulic jump on currents of these fluids must be carefully considered, as an increase in flow depth associated with the hydraulic jump can cause overspill of toxic or highly polluting substances out of channel boundaries and associated contamination. In the following, the equations describing the phenomenon are derived and an approximate analytical solution for the critical depth is proposed.

2 THEORY

2.1 Channel geometry

A horizontal, wide rectangular channel with negligible wall effects, in which a Herschel-Bulkley fluid flows, is subject to an abrupt deflection of its axis by an angle θ, causing a perturbation in the flow conditions, as shown in Figure 1a. The change in flow conditions occurs only downstream of a shock wave, originating in the vertex of the deflection and inclined by an angle β from the wall; β is called the Mach angle.

The flow is laminar, and the velocity distribution is shown in Figure 1b and in equation (1), derived by *Di Federico* (1998); in (1), Y is the flow depth, U_0 is the shear-free velocity, $\xi = y/Y$ and $\lambda = \tau_0/\tau_w$ the ratio between the yield stress and the bottom shear stress.

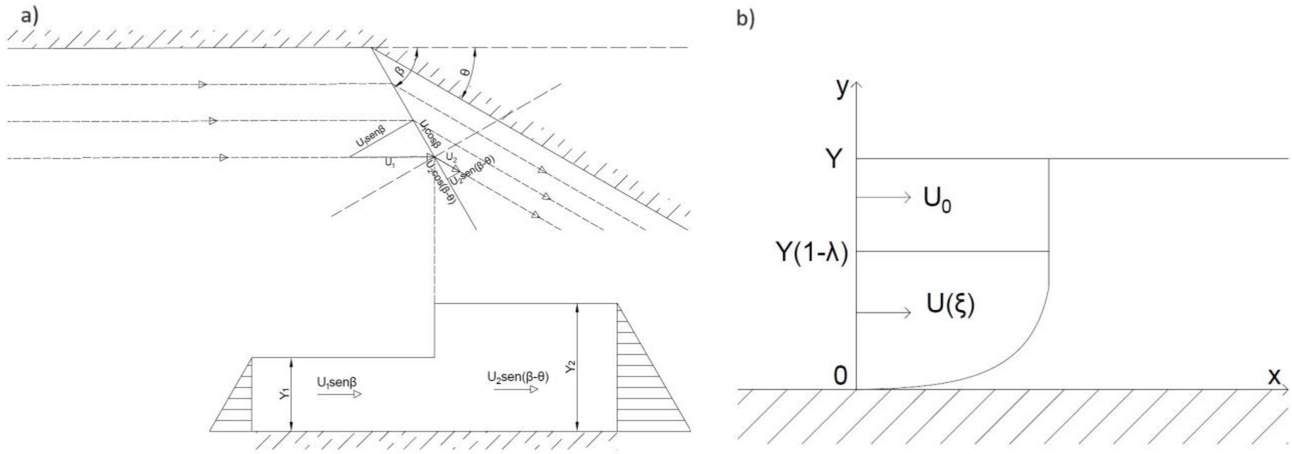

Figure 1. a) Geometry of the channel, b) Velocity profile

$$\begin{cases} U(\xi) = U_0 \quad \text{for} \quad 1-\lambda \leq \xi \leq 1 \\ U(\xi) = U_0\left[1-\left(\dfrac{1-\lambda-\xi}{1-\lambda}\right)^{\frac{n+1}{n}}\right] \quad \text{for} \quad 0 \leq \xi \leq 1-\lambda \end{cases} \tag{1}$$

2.2 Governing equations

A Herschel-Bulkley fluid follows the three-parameter model shown in equation (2), where U is the velocity, τ is the shear stress, τ_0 is the yield stress, K is the consistency and n is the flow behaviour index (*Longo et al.*, 2015).

$$\begin{cases} \tau = \tau_0 sgn\left(\dfrac{\partial U}{\partial y}\right) + K\left|\dfrac{\partial U}{\partial y}\right|^{n-1}\dfrac{\partial U}{\partial y} \quad \text{for} \quad |\tau| > \tau_0 \\ \dfrac{\partial U}{\partial y} = 0 \quad \text{for} \quad |\tau| < \tau_0 \end{cases} \tag{2}$$

To describe the phenomenon, four equations have to be solved: the mass balance equation, the momentum balance equations in both directions, perpendicular and parallel to the wave front, and the constitutive equation for a Herschel-Bulkley fluid.

The solution of these equations gives a system of four transcendental equations (3) with four unknowns which can only be solved by numerical methods, where $\eta = Y_2/Y_1$ is the ratio between downstream and upstream depths and $\zeta = U_{02}/U_{01}$ the respective velocity ratio. Known parameters are the geometry of the channel, the rheological properties of the fluid and the upstream flow conditions, which can be expressed by the Froude number Fr_1.

$$\begin{cases} \zeta\eta\dfrac{\sin(\beta-\theta)}{\sin(\beta)} = \dfrac{1+n+n\lambda_1}{1+n+n\lambda_2} \\ \dfrac{(1-\eta^2)}{2Fr_1^2}\dfrac{(1+n+n\lambda_1)^2}{2n+1} = \eta\zeta^2\dfrac{2(n+1)^2+n\lambda_2(4n+3)}{3n+2}sin^2(\beta-\theta) - \dfrac{2(n+1)^2+n\lambda_1(4n+3)}{3n+2}sin^2(\beta) \\ \left(\dfrac{\eta}{\zeta}\right)^n = \dfrac{\lambda_2}{\lambda_1}\left(\dfrac{1-\lambda_1}{1-\lambda_2}\right)^{n+1} \\ \eta = \dfrac{\tan(\beta)}{\tan(\beta-\theta)} \end{cases} \tag{3}$$

It can be observed that the axis deflection θ causes an increase of the downstream depth, which increases for larger values of θ, as shown in Figure 2. The deflection also forces the downstream velocity to decrease. Note that, since $0 < \beta < \pi/2$, there is an inverse relationship between the Mach angle and the upstream Froude number. This means that for a flow which is well into the supercritical regime, the hydraulic jump is located at a greater distance from the deflection section.

From the general case for a Herschel-Bulkley fluid, other special cases can be derived. Substitution of the rheological parameters n and τ_0 in the model, gives simplified systems for Bingham, Ostwald-DeWaele and Newtonian fluids.

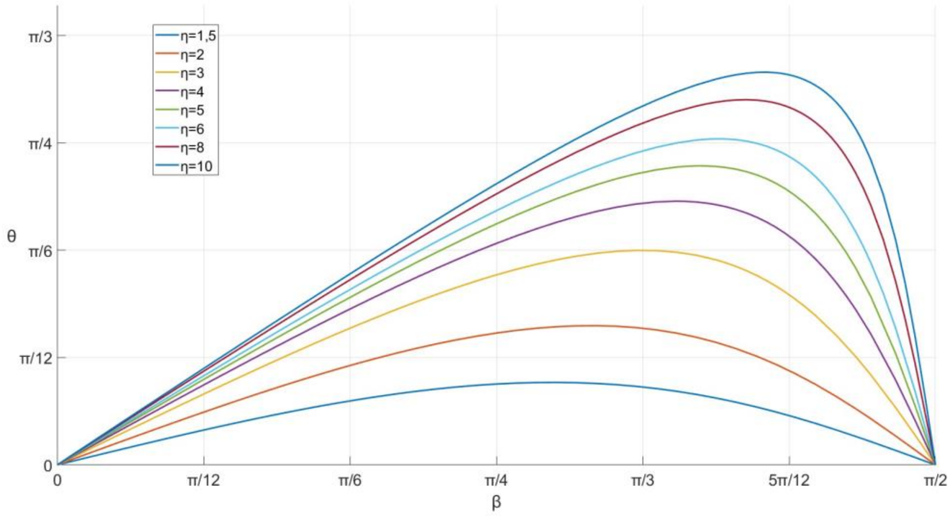

Figure 2. Deflection θ versus Mach angle β for various values of depth ratio η

Since the system is only solvable by numerical methods, an approximate analytical solution is also sought, following *Zhou et al.* (2007) and *Ugarelli & Di Federico* (2007). This solution is obtained from an expansion in Taylor series, which gives a non-algebraic formula allowing the direct calculation of the downstream depth. With further simplifications, an approximate value for the critical depth is obtained.

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